Fair and Efficient Allocation of Indivisible Public Goods

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(Work done as a part of Game Theory & Mechanism Design Course Term Project)

April 10, 2023
E1 - 254

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Introduction

- The concept of fair allocation of goods was first proposed by [10], and since then, it has gained huge popularity with applications in several fields.
- The problem revolves around how to distribute resources among a group of agents in a way that is deemed fair by all participants.
- Further goods can be either divisible or indivisible. Divisible goods are those that can be divided into smaller portions, such as money or food, while indivisible goods are those that cannot be divided, such as a house or a car. The division of divisible goods is relatively straightforward [3], while the division of indivisible goods is much more complex [1], as each agent must be allocated an entire unit of the resource.
- Further, we can broadly divide goods into two categories based on the distribution of resources: private and public.
- A personal item, for example, a car, is a private good that only has value to the agent it is assigned to.
- A public good, on the other hand, like a book in a library or a park, can benefit numerous agents at once. The distribution of public goods presents specific difficulties because it significantly affects the welfare of all agents involved.
- In this presentation we will look into models of efficient and fair allocation of public goods and its connection with private goods.

Definitions and Terminologies: Problem Settings

Consider $\mathcal{A} = [n]$ is a set of n agents and $\mathcal{G} = [m]$ is a set of m goods.

PrivateGoods. A private good (PRIVATEGOODS) instance can be defined as a tuple $(\mathcal{A}, \mathcal{G}, \mathcal{V})$. $\mathcal{V} = \{v_i\}_{i \in A}$ a set of utility function for each agent.

- An allocation $x = (x_1, ...x_n) \in \prod_n \mathcal{G}$ is a n partition of the goods ($\mathcal{G}$) into n parts $x_1 ... x_n$, where agent i is assigned the bundle x_i and thus gets the utility of $v_i(x_i)$.

PublicGoods. A public good (PUBLICGOODS) instance can be defined as a tuple $(\mathcal{A}, \mathcal{G}, k, \mathcal{V})$. Out of m goods at most k can be selected. $\mathcal{V} = \{v_i\}_{i \in A}$ a set of utility function for each agent.

- An allocation x is a subset of G of size at most k, $(|x| \leq k)$, giving agent i an utility of $v_i(x)$
- Examples of $k < n$ (well studied) : voting and $k > n$ (not well studied) : Books in a library

PublicDecisions. A public decision (PUBLICDECISION) instance first proposed in [4] can be defined as a tuple $(\mathcal{A}, \mathcal{G}, \mathcal{V})$
$\mathcal{G} = [m]$ is a set of m issues and each $j \in \mathcal{G}$ has a set of k_j alternatives defined as $G_j := (j, 1)...(j, k_j)$, $\mathcal{V} = \{v_i\}_{i \in A}$ a set of utility functions for each agent where agent i has the value $v_i(j, l)$ for the lth alternative of the jth issue.

- An allocation $x = (x_1 ... x_m)$ comprises of m decisions where $x_j \in [k_j]$ is the decision on issue j thereby giving agent i an utility $v_i(x) = \sum_{j \in \mathcal{G}} v_i(j, x_j)$

Definitions and Terminologies: Fairness

- **Proportionality (Prop) and α-Proportionality (α-Prop):** This fairness notion ensures every agent should receive its proportion of goods available. Thus, the proportional share of agent i is denoted as $Prop_i = \frac{v_i(\mathcal{G})}{n}$. We say an allocation satisfies α-proportionality if $v_i(x) \geq \alpha Prop_i \ \forall i$.
- **Relaxation of α-Proportionality upto 1 good (α-Prop1):** This fairness notion ensures every agent gets its prop by swapping atmost one good from his allocation with a good outside its allocation. Formally, this be be defined as: An allocation x is Prop1 if $\exists g \in x, g' \in \mathcal{G}$ such that $v_i((x - g) \cup g') \geq \alpha Prop_i \ \forall i$.
- Its evident that Prop1 ensures fairness at an individual level and is less strong than Prop.
- **Envy-freeness:** This fairness notion ensures every agent i should prefer their own allocation over any other agent j's allocation, in a sense the agents dont envy each other.
- **Pareto-dominance** An allocation y pareto - dominates x if $\forall i \in \mathcal{A}, v_i(y) \geq v_i(x)$ (one of the inequalities must be strict). An allocation x is called Pareto-optimal if there exists no allocation that Pareto-dominates x. i.e, It would be impossible to make agent i better without making another agent j worse.

Definitions and Terminologies: Allocations

Nash welfare (NW) can be defined the geometric mean of agents utilities, i.e.,

$$NW(x) = (\prod_{i \in A} v_i(x))^{1/n}$$

NW allocation ensures fairness in allocation of goods [9]. MNW allocations (that maximizes NW) are *good* in the sense that they are pareto-optimal and fair (satisfies relaxations of envy-freeness and proportionality for PrivateGoods and PublicDecision)

The MNW allocations for the three problem setting are as follows:

- PrivateGoods: $\text{argmax}_{x \in \prod_n(\mathcal{G})} NW(x)$
- PublicGoods : $\text{argmax}_{x \subseteq \mathcal{G}, |x| \leq k} NW(x)$
- PublicDecision : $\text{argmax}_{x \in decisions} NW(x)$

Another *good allocation* is lexmin.

- The lexmin allocation is *good* as it can be thought of a mechanism that first maximizes the minimum utility that any agent gets followed maximising the second lowest utility and so on. The lexmin allocation satisfies the fairness notions of proportionality, envy freeness along with pareto-optinality [8].

Main results of [7]

[7] addresses three main questions which we shall discuss briefly next, these three questions are as follows:

- Relating the three problem setting PrivateGoods, PublicGoods, and PublicDecision
- Fairness and Efficiency guarantees in PublicGoods
- Computing the computational complexities of the MNW and Lexmin allocations for PublicGoods

Relating the three problem setting

Theorem. (1) PrivateMNW polynomial-time reduces to PublicMNW (2) PublicMNW polynomial-time reduces to DecisionMNW

Given an instance of PrivateGoods $\mathcal{I} = (\mathcal{A} = [n], \mathcal{G} = [m], V)$ construct an instance of PublicGoods $\mathcal{I}' = (\mathcal{A}' = [m+n], \mathcal{G}' = [m.n], k = m, \mathcal{V}')$

- Create agents in $\mathcal{I}$' corresponding to the agents in $\mathcal{A}$. Then add m dummy agents, one corresponding to each private good.
- Now introduce $m.n$ public goods by making n copies of each good in $\mathcal{G}$. Set k as the number of private goods
- Define the valuations of agent i in $\mathcal{A}'$ as follows:
- Each agent i values the ith copy of every good j at what they valued them at the private good setting, and values all the other goods at zero.
- Each dummy agent j which is created corresponding to the jth private good values all public good copies of jth good at value 1 and 0 otherwise.

Relating the three problem setting

Now the reduction follows as:

- Suppose MNW allocation of $\mathcal{I}$' has positive nash welfare, so each agent gets positive utility.
- Now note that the jth dummy agent only values copies of the jth good and since they have positive utility so they get atleast one copy of the jth good.
- But, we have $k = m$, so only m public goods can be selected in an allocation and there are m dummy agent. So by pigeon hole principle exactly one copy of each good is selected in the allocation.
- Now from an allocation x' of $\mathcal{I}$' we can construct an allocation x of I. Observe that the ith agent only values the ith copy of goods.
- Thus x can be constructed as $x_i = \{j \in G : j_i \in x'\}$. Clearly then $v_i(x_i) = v'_i(x')$ and every dummy agent d gets value $v'_d(x') = 1$.
- Thus product of utilities of agents in $x =$ product of utilities of agents in x'. (nash product)
- The same argument holds in the other direction by reversing the logic. i.e, from an private goods instance and an allocation of private goods, we can create a public goods instance and an allocation of public goods with the same product of utilities (nash product).

Fairness and Efficiency

Pareto-Optimality. Suppose MNW allocations do not satisfy pareto optimality, this would mean one of agent could get a strictly higher value keeping the values of other agents non decreasing.
Consider two cases:

- MNW value $\neq 0$: Then we can get an allocation whose NW is greater. Contradiction.
- MNW value $= 0$: If the value increase holds for an agent with non zero value initially, the nash product over these agents increases, contradiction. Else, if the value increases for an agent with zero value initially then the number of agents with non zero values increases, again a contradiction to optimality of MNW.

Theroem. The MNW allocations for PublicGoods satisfy Prop1, Pareto-Optimality, and $1/n-$RRS. Further when $k \geq n$, MNW allocation satisfies $\frac{1}{2n-1}-$Prop.

Theroem. The Lexmin allocations satisfy Prop1, Pareto-Optimality. Further when $k \geq n$, lexmin allocation satisfies $\frac{n}{2n-1}-$Prop.

Computational Complexity

- The problem for computational complexity of PrivateGoods are well studied in literature. [5, 2] has shown PrivateMNW is $\mathcal{NP}$-hard for $N = 2$ agents.
- However the PrivateMNW problem becomes solvable in polynomial time if the valuations are binary [5, 2].
- PublicMNW is $\mathcal{NP}$-hard for $k < n$ (using reduction from PrivateMNW)
- In fact, [7] shows PublicMNW ($k \geq n$)is $\mathcal{NP}$-hard even when:
    - Valuations are binary
    - There are only two agents
- Based on the reduction from PublicMNW to DecisionMNW, DecisionMNW is also $\mathcal{NP}$-hard. (Even when valuations are binary)
- Similar complexity are for PublicLex and DecisionLex

The proofs are based on a reduction of PublicMNW to the Exact Regular Set Packing (ERSP) problem.
Given n elements in $X = (x_1, x_2, ..., x_{n)}$ and a family of subsets of X, $F = \{F_1, ..F_m\}$ with $|F_j| = d$, the problem is to compute a subfamily $F' \subseteq F, |F'| = r$ such that $\forall F_i \neq F_j \in F', F_i \cap F_j = \{\}$. The ERSP problem is NP-hard [6].

My ideas: PublicGoods-Cost

We extend the given definition of a public good to a scenario where each good has a cost and agents have a collective budget.

Definition. PublicGoods-Cost
A public good with cost (PublicGoods-Cost) instance can be defined as a tuple $(\mathcal{A}, \mathcal{G}, k, \mathcal{V}, \mathcal{C}, B)$, where:

- $\mathcal{A} = [n]$ is a set of n agents
- $\mathcal{G} = [m]$ is a set of m goods, where each good $j \in \mathcal{G}$ has a cost $c_j \in \mathbb{R} \geq 0$
- $k \in \mathbb{Z}_{\geq 0}$ is the maximum number of goods that can be selected
- $\mathcal{V} = (v_i)_{i \in A}$ a set of agent utility functions, where $v_i : 2^{\mathcal{G}} \to \mathbb{R} \geq 0$ is a function that maps a set of goods to the utility that agent i derives from it
- $\mathcal{C} = (c_j)_{j \in \mathcal{G}}$ is the set of costs of the goods
- $B \in \mathbb{R}_{\geq 0}$ is the collective budget of the agents

An allocation x is a subset of $\mathcal{G}$ of size at most k, ($|x| \leq k$), giving agent i a utility of $v_i(x)$, subject to the budget constraint $\sum_{j \in x} c_j \leq B$.

My ideas: PublicGoods-Cost

- This model reflects the fact that the agents want to find an allocation or find a set of goods that maximizes their utility but subject to the constraint that the total cost of the selected goods do not exceed a collective budget.
- **Example:**
- Consider a Public Transit System where there are n commuters and m origin-destination pairs for routes.
- Each route has a different cost associated (some are longer and need more fuel), c_j.
- Suppose the transit authority has total collective budget as B.
- The objective is to maximize the total utility of the commuters, which may depend on several factors such as the travel time, safety or ease of the travel.
- The company needs to find maximum k OD pairs in a city.
- Clearly the model can be defined in a PublicGoods-Cost setting.

Allocation. The MNW allocations for PublicGoods-Cost is Pareto Optimal. The MNW allocation for this problem can be defined as follows:

$$x^* = argmax_{x \subseteq G, |x| \leq k, \sum_{j \in x} c_j \leq B} \prod_{i \in A} v_i(x)$$

Fairness.

- $\alpha - Prop1$ for the MNW allocation can be defined as follows

$$\exists g \in x, g' \in G \text{ with } \sum_{j \in (x-g) \cup g'} c_j \leq B$$

such that

$$v_i((x-g) \cup g') \geq \alpha . Prop_i \ \forall i$$

- $\alpha - Prop1$ doesn't balance the trade off between the valuations and cost of goods, hence we can define a new fairness notion that ensures the difference between the total value derived from the goods by a group of agents and the total cost of goods is divided fairly among the agents of that group which is a stronger fairness notion.
- Let's define this fairness notion as Cost Adjusted Proportionality or CAP. We say an allocation x is CAP if:

$$\frac{v_i(x)}{\sum_{j \in x} c_j} \geq \beta \frac{Prop_i}{\sum_{j \in G} c_j} \ \forall i$$

My ideas: PublicGoods-Cost

Theorem. The computation of MNW allocations for PublicGoods-Cost is NP Hard even for binary valuations.

- We show this by doing a reduction from PublicGoods-Cost problem with budget constraints to the PublicGoods problem without budget constraints.
- We will show that an instance of PublicGoods problem can be transformed into an instance of the PublicGoods-Cost problem.
- Consider an instance of the PublicGoods problem, described by the tuple $(\mathcal{A}, \mathcal{G}, k, \mathcal{V})$
- We can transform this instance into an instance of the PublicGoods-Cost problem by introducing budget constraints. Define the tuple $(\mathcal{A}, \mathcal{G}, k, \mathcal{V}, \mathcal{C}, B)$, where:
- $\mathcal{A}, \mathcal{G}, k, \mathcal{V}$ remain the same as in the original PublicGoods problem
- $\mathcal{C} = c_{j_{j \in \mathcal{G}}}$ is the set of costs of the goods, where each cost $c_j = 1$ for all $j \in \mathcal{G}$, that is cost of each good is 1.
- Set $B = k$. That is the collective budget of the agents is set equal to the maximum number of goods that can be selected.
- Now, observe that any allocation of goods in the PublicGoods problem can also be an allocation in the PublicGoods-Cost problem
- Therefore, any solution to the PublicGoods problem is also a valid solution for the corresponding PublicGoods-Cost problem. This shows that the PublicGoods-Cost problem is NP-hard.

Summary

- We studied fair and efficient allocation of indivisible public goods.
- [7] showed the maximum nash welfare and lexmin allocation for the setting are efficient and fair, and that computing them is NP hard.
- The authors also presented polynomial time reductions between the MNW and lexmin allocations in the private good, public good and public decision setting.
- We extended the Public goods formulation to a Public goods with cost instance, where each public good has some costs and there are budget constraints.
- We showed the MNW allocation for the problem and proposed a new fairness notion and an allocation based on that as a multi objective optimisation problem.
- We showed solving the MNW allocation for this problem is NP hard.
- We also proposed a naive greedy heuristic to get a good allocation for the problem.
- There are several future directions for this setting – investigating if the reductions to public decisions will hold and if the proposed MNW allocation in this setting fair.
- Other future directions include developing a polynomial time prop1 and pareto-optinal allocation for public goods.

Thank You!

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