# Fair and Efficient Allocation of Indivisible Public Goods

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## Abstract

In this project, we will study the fair and efficient allocation of public goods. We will mainly study the recent results by [8]. The main contributions of [8] are to establish connections between three models of good allocation - Public Good, Private Good and Public Decision and present polynomial time reductions between these models for Maximum Nash Welfare (MNW) and leximin allocations. [8] also shows MNW and lexmin allocations for public goods are fair and efficient. Finally, the authors present the computation complexity of these allocations and approximation algorithms for the same. This write-up summarises the works of [8] and related papers very briefly.

The concept of fair allocation of goods was first proposed by [11], and since then, it has gained huge popularity with applications in several fields, including mathematics, computer science, and economics, owing to its real-life applications, such as the distribution of goods and services, the split of assets in divorce settlements, or the division of family responsibilities. The problem revolves around how to distribute resources among a group of agents in a way that is deemed fair by all participants. Thus, fair division is not only a theoretical issue; it also has several practical uses.

Further, [8] has the constraint that goods are indivisible. divisible goods are those that can be divided into smaller portions, such as money or food, while indivisible goods are those that cannot be divided, such as a house or a car. The division of divisible goods is relatively straightforward [4], while the division of indivisible goods is much more complex [1], as each agent must be allocated an entire unit of the resource.

We can broadly divide goods into two categories based on the distribution of resources: private and public. A personal item, for example, a car, is a private good that only has value to the agent it is assigned to. A public good, on the other hand, like a book in a library or a park, can benefit numerous agents at once. The distribution of public goods presents specific difficulties because it significantly affects the welfare of all agents involved. The paper [8] focuses on models of public good allocation and its connection with private goods. A private good (PRIVATEGOODS) instance can be defined as a tuple $(\mathcal{A}, \mathcal{G}, \mathcal{V})$. where $\mathcal{A} = [n]$ is a set of $n$ agents, $\mathcal{G} = [m]$ is a set of $m$ private goods and set $\mathcal{V} = \{v_i\}_{i \in A}$ a set of utility function for each agent, where the function $v_i$ tells us the utility that agent $i$ has for a subset of the goods. Throughout the paper, the authors have assumed the valuation functions are additive. An allocation $x = (x_1, ...x_n) \in \prod_n \mathcal{G}$ is a $n$ partition of the goods ($\mathcal{G}$) into $n$ parts $x_1...x_n$, where agent $i$ is assigned the bundle $x_i$ and thus gets the utility of $v_i(x_i)$. Private goods have been well studied theoretically [9]. The paper majorly focuses on two more models: Public Goods (PUBLICGOODS) and Public Decisions (PUBLICDECISIONS).

A public good (PUBLICGOODS) instance can be defined as a tuple $(\mathcal{A}, \mathcal{G}, k, \mathcal{V})$, where $\mathcal{A} = [n]$ is a set of $n$ agents, $\mathcal{G} = [m]$ is a set of $m$ private goods out of which at most $k$ can be selected and set $\mathcal{V} = \{v_i\}_{i \in A}$ a set of utility function for each agent. An allocation $x$ is a subset of $G$ of size at most $k$. The case of $k < n$ is similar to committee selection or voting and is well studied in the literature, whereas the case of $k \geq n$ isn't studied much [8]. The paper provides several motivating examples for the case of $k \geq n$ like $k$ books in a public library of $n$ readers. [8] points out it is important to ensure fairness at an individual level while allocating public goods. The final model, A public decision (PUBLICDECISION) [6] instance can be defined as a tuple $(\mathcal{A}, \mathcal{G}, \mathcal{V})$, where $\mathcal{A} = [n]$ is a set of $n$ agents, $\mathcal{G} = [m]$ is a set of $m$ issues and each $j \in \mathcal{G}$ has a set of $k_j$ alternatives. The set $\mathcal{V} = \{v_i\}_{i \in A}$ a set of utility functions for each agent where agent $i$ has the value $v_i(j, l)$ for the $l$th alternative of issue $j$. An allocation $x = (x_1...x_m)$ comprises of $m$ decisions where $x_j \in [k_j]$ is the decision on issue $j$.

A central solution concept to fairly allocate goods is the Nash Welfare which is simply the geometric mean of agent utilities [10]. The Maximum Nash Welfare problem (MNW) is to find the allocation to maximise nash welfare [10]. The paper labels these allocations for the three problems discussed above as PRIVATEMNW, which searches over all partitions of the goods into $n$ parts, PUBLICMNW which searches over the space of all subsets of goods size at most $k$ and DECISIONMNW which looks into all decisions on the allocations. Similar to MNW allocations, the authors of [8] also study the lexmin allocation (a allocation is lexmin optimal if no other allocation lexmin-dominates it. [8]). These allocations are similarly labelled PRIVATELEX, PUBLICLEX and DECISIONLEX. MNW and Lexmin allocations are a good allocation solution as these give efficient Pareto optimal (PO) solutions and is fair as it satisfies envy-freeness and proportionality for PRIVATEGOODS and PUBLICDECISION [5, 6]. The paper investigates how good MNW and Lexmin is for PUBLICGOODS. An allocation $x$ is called proportional if every agent gets at least one $n$th share of our value of all the goods, but proportional alocations don't always exist, hence a relaxation of proportionality upto one good (called Prop1) is used. Another fairness notion called Round Robin Share ($RRS_i$) that denotes the minimum value that an agent is guaranteed if the agents pick $k$ goods in round robin fashion, $i$ picking at last. [8] showed that all MNW and Lexmin allocations for PUBLICGOODS are PO, Prop1 and 1/n-RRS, thereby showing these allocations are fair and effitient. The authors also showed for $k \geq n$ the allocation implies proportionality, which is a strong fairness notion. [3].

Further the authors of [8] investigate connections between the three mdoels: PRIVATEGOODS, PUBLICGOODS and PUBLICDECISION. Here the authors show there are polynomial time reductions between these. The paper presents novel polynomial time reduction from PRIVATEGOODS to PUBLICGOODS and from PUBLICGOODS to PUBLICDECISION for the MNW and Lexmin mechanisms with PRIVATEMNW $\leq$ PUBLICMNW $\leq$ DECISIONMNW and PRIVATELEX $\leq$ PUBLICLEX $\leq$ DECISIONLEX.

After proving that MNW and Lexmin allocations for public goods are fair and efficient, finally, the authors of [8] prove the computational complexity of PUBLICMNW and PUBLICLEX. The problem is well studied for PRIVATEGOODS. [7, 2] has shown PRIVATEMNW is $\mathcal{NP}$-hard for $N = 2$ agents, but the problem becomes solvable in polynomial time if the valuations are binary [7, 2]. Using the reduction from PRIVATEGOODS to PUBLICGOODS, the authors of [8] show PUBLICMNW is $\mathcal{NP}$-hard for $k < n$. The authors also show PUBLICMNW is $\mathcal{NP}$-hard for any $k$ and even for binary valuation which is in contrast to the PRIVATEMNW case. Finally using the reduction from PUBLICGOODS to PUBLICDECISION, the authors of [8] show DECISIONMNW is $\mathcal{NP}$-hard even for binary valuations. Thus to calculate PUBLICMNW in $\mathcal{O}(n)$ time, the authors propose an approximation algorithm for $k \geq n$ for agents with monotone, subadditive utility function which is a generalisation of additive utilities. The authors also show there exists a pseudo-polynomial time exact algorithm for PUBLICMNW with runtime $\mathcal{O}((m * max_{i,j} v_i v_j)^n)$ when the numbers of agents are constant.

Thus, the main contributions of the [8] are to show MNW and Lexmin allocations for PUBLICGOODS are effitient and fair and that computing PUBLICMNW and DECISIONMNW are $\mathcal{NP}$-hard even for binary valuation case. The authors give polynomial time reductions between the PRIVATEGOODS, PUBLICGOODS and PUBLICDECISION models and finally gives and approximation algorithms for computing MNW and Lexmin for PUBLICGOODS.

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