Fair and Efficient Allocation of Indivisible Public Goods

Debojjal Bagchi
Undergraduate
Indian Institute of Science, Bengaluru
SRN : 16849
debojjalb@iisc.ac.in

Abstract

In this report, we study the fair and efficient allocation of indivisible public goods. We will mainly study the recent results by [18]. The main contributions of [18] are to establish connections between three models of good allocation - Public Good, Private Good and Public Decision and present polynomial time reductions between these models for Maximum Nash Welfare (MNW) and leximin allocations. [18] also shows MNW and lexmin allocations for public goods are fair and efficient. Finally, the authors present the computation complexity of these allocations and approximation algorithms for the same. In this report a brief survey of the work related to the three problem setting is presented and further extensions are discussed.

1 Introduction

The concept of fair allocation of goods was first proposed by [23], and since then, it has gained huge popularity with applications in several fields, including mathematics, computer science, and economics, owing to its real-life applications, such as the distribution of goods and services, the split of assets in divorce settlements, or the division of family responsibilities. The problem revolves around how to distribute resources among a group of agents in a way that is deemed fair by all participants. Thus, fair division is not only a theoretical issue; it also has several practical uses.

Further goods can be either divisible or indivisible. Divisible goods are those that can be divided into smaller portions, such as money or food, while indivisible goods are those that cannot be divided, such as a house or a car. The division of divisible goods is relatively straightforward [10], while the division of indivisible goods is much more complex [2], as each agent must be allocated an entire unit of the resource.

Finally, we can broadly divide goods into two categories based on the distribution of resources: private and public. A personal item, for example, a car, is a private good that only has value to the agent it is assigned to. A public good, on the other hand, like a book in a library or a park, can benefit numerous agents at once. The distribution of public goods presents specific difficulties because it significantly affects the welfare of all agents involved. The paper [18] focuses on models of public good allocation and its connection with private goods.

In this study, we look into efficient and fair allocation of indivisble public goods. The rest of the report is organised as follows: Section 2 discussed various definitions and terminologies, Section 3 goes over some of the related work, Section 3 details the main results of [18] along with brief proof sketches, 5 extends the work in a new direction and 6 concludes the discussion and gives future research directions.

2 Definitions and Terminologies

2.1 The three problem settings

Definition. PRIVATEGOODS

A private good (PRIVATEGOODS) instance can be defined as a tuple $(\mathcal{A}, \mathcal{G}, \mathcal{V})$. where:

- $\mathcal{A} = [n]$ is a set of n agents
- $\mathcal{G} = [m]$ is a set of m private goods
- $\mathcal{V} = \{v_i\}_{i \in A}$ a set of utility function for each agent, where the function $v_i : 2^G \to \mathcal{R}_{\geq 0}$ is the utility that agent i has for a subset of the goods. Typically, additivity is assumed as follows: $v_i(S) = \sum_{j \in S} v_i(j)$

An allocation $x = (x_1, ...x_n) \in \prod_n \mathcal{G}$ is a n partition of the goods $(\mathcal{G})$ into n parts $x_1...x_n$, where agent i is assigned the bundle x_i and thus gets the utility of $v_i(x_i)$.

Definition. PUBLICGOODS

A public good (PUBLICGOODS) instance can be defined as a tuple $(\mathcal{A}, \mathcal{G}, k, \mathcal{V})$, where:

- $\mathcal{A} = [n]$ is a set of n agents
- $\mathcal{G} = [m]$ is a set of m private goods out of which at most k can be selected
- $\mathcal{V} = \{v_i\}_{i \in A}$ a set of agent utility functions, where $v_i : 2^G \to \mathcal{R}_{\geq 0}$

An allocation x is a subset of G of size at most k, $(|x| \leq k)$, giving agent i an utility of $v_i(x)$

Insights. k can be interpreted as a public budget that an group of agent has which they want to use collectively on public goods which are unit priced each. Thus, the model is closely related to participatory budgeting. In simple terms: n agents wants to select k public goods from m available public goods. Examples for $k < n$ includes voting, committee selection, etc. For the case of $k \geq n$, a motivating example would be k books in a public library of n readers. It is important to ensure fairness at an individual level while allocating public goods. For instance, in the library-book example, a good allocation should incorporate *taste* of each reader.

Definition. PUBLICDECISIONS

A public decision (PUBLICDECISION) instance first proposed in [13] can be defined as a tuple $(\mathcal{A}, \mathcal{G}, \mathcal{V})$, where:

- $\mathcal{A} = [n]$ is a set of n agents
- $\mathcal{G} = [m]$ is a set of m issues and each $j \in \mathcal{G}$ has a set of k_j alternatives defined as $G_j := (j, 1)...(j, k_j)$
- $\mathcal{V} = \{v_i\}_{i \in A}$ a set of utility functions for each agent where agent i has the value $v_i(j, l)$ for the lth alternative of the jth issue. The valuations are additive.

An allocation $x = (x_1...x_m)$ comprises of m decisions where $x_j \in [k_j]$ is the decision on issue j thereby giving agent i an utility $v_i(x) = \sum_{g \in \mathcal{G}} v_i(j, x_j)$

Insights. As in the case of PUBLICGOODS a decision made in PUBLICDECISION is expected to be *fair* to the participants.

2.2 Quantifying Fairness

As seen in section 1.1 it is important ti ensure fairness when dividing goods among agents. However maintaining fairness can be challenging as it competes with efficiency [8]. The following subsection discusses some of the popular fairness notions [20, 8].

- **Proportionality (Prop) and α-Proportionality (α-Prop):** This fairness notion ensures every agent should receive its proportion of goods available. Thus, the proportional share of agent i is denoted as $Prop_i = \frac{v_i(\mathcal{G})}{n}$. We say an allocation satisfies α-proportionality if $v_i(x) \geq \alpha Prop_i \ \forall i$. We say just Prop if $\alpha = 1$. It should be noted that Proportionality satisfying allocations might not always exist. Finally, its evident that Prop ensures fairness at an individual level.

- **Relaxation of α-Proportionality upto 1 good (α-Prop1):** This fairness notion ensures every agent gets its prop by swapping atmost one good from his allocation with a good outside its allocation. Formally, this be be defined as: An allocation x is Prop1 if $\exists g \in x, g' \in \mathcal{G}$ such that $v_i((x - g) \cup g') \geq \alpha Prop_i \ \forall i$. We say just Prop1 if $\alpha = 1$. Again, its evident that Prop1 ensures fairness at an individual level and is less strong than Prop.
- **α-Round Robin Share ($\alpha - RRS$):** The round-robin share of agent i, denoted by RRS_i is the minimum value an agent can be guaranteed if the agents pick k goods in a round-robin fashion, with i picking last. An allocation x is said to be $\alpha - RRS$ if $\forall i \in \mathcal{A}, v_i(x) \geq \alpha RRS_i$
- **Envy-freeness:** This fairness notion ensures every agent i should prefer their own allocation over any other agent j's allocation, in a sense the agents dont envy each other.
- **Pareto-optimality:** This fairness notion ensures that it should be impossible to make an agent i better without making another agent $j \neq i$ worse. Formally, an allocation y is said to *Pareto-dominate* an allocation x if $\forall i \in \mathcal{A}, v_i(y) \geq v_i(x)$ (one of the inequalities must be strict). An allocation x is called Pareto-optimal if there exists no allocation that Pareto-dominates x.

2.3 Allocation Strategies

Strategy 1: Maximum Nash Welfare (MNW) allocation

Nash welfare (NW) can be defined the geometric mean of agents utilities, i.e.,

$$NW(x) = (\prod_{i \in A} v_i(x))^{1/n}$$

NW allocation ensures fairness in allocation of goods [21].

An allocation that maximises the Nash Welfare is referred to as a Maximum Nash Welfare (MNW) Allocation. This allocation helps to balance between maximising the sum of utilities (utilitarian social welfare) and individual fairness [3]. MNW allocations are *good* in the sense that they are pareto-optimal (no dominating allocations) and fair (satisfies relaxations of envy-freeness and proportionality for PRIVATEGOODS and PUBLICDECISION)

The MNW allocations for the three problem setting are as follows:

- PRIVATEGOODS: $\text{argmax}_{x \in \prod_n (\mathcal{G})} NW(x)$
- PUBLICGOODS : $\text{argmax}_{x \subseteq \mathcal{G}, |x| \leq k} NW(x)$
- PUBLICDECISION : $\text{argmax}_{x \in decisions} NW(x)$

The MNW allocations for PRIVATEGOODS, PUBLICGOODS, and PUBLICDECISION are denoted as PRIVATEMNW, PUBLICMNW, and DECISIONMNW

Strategy 2: lexmin allocation

Given an allocation x, let x_{lex} denote the vector of agent's utilities under x, sorted in non-decreasing order. For two allocations x and y, x *leximin-dominates* y if:

- $\exists i \in [n]$ s.t. $x_{lex} > y_{lex}$
- $x_{lex} = y_{lex} \ \forall j < i$.

An allocation is leximin-optimal if no other allocation leximin-dominates it.

The lexmin allocation is *good* as it can be thought of a mechanism that first maximizes the minimum utility that any agent gets followed the second lowest utility. Further the lexmin allocation satisfies the fairness notions of proportionality, envy freeness along with pareto-optinality [20].

The lexmin allocations for PRIVATEGOODS, PUBLICGOODS, and PUBLICDECISION are denoted as PRIVATELEXMIN, PUBLICLEXMIN, and DECISIONLEXMIN

3 Related Works

Fair division of goods have been in study for past 6 decades. This line of research was first started by [24] by the cake cutting problem. In case of division of goods several fairness notions have been developed. Among these, proportionality and envy freeness introduced by [24, 17] are most famous. However these are too strong to be delivered in practical scenarios [7].. Hence several relaxed fairness notions are proposed in literature, which have been discussed in earlier section.

Allocation of Private goods have been well studied theoretically [19]. A central solution concept to fairly allocate goods is the Nash Welfare which is simply the geometric mean of agent utilities [21]. The Maximum Nash Welfare problem (MNW) is to find the allocation to maximise nash welfare [21]. Similar to MNW allocations, the authors of [18] also study the lexmin allocation. MNW and Lexmin allocations are a good allocation solution as these give efficient Pareto optimal (PO) solutions and is fair as it satisfies envy-freeness and proportionality for PRIVATEGOODS and PUBLICDECISION [11, 13].

[1] studied the problem of allocating maxmin share allocation, a recently introduced fairness notion. [4] also looked into maxmin allocation in respect of indivisible goods. [9] investigated several different fairness notion for allocation of indivisible goods. There has been extensive studies related to maximum nash welfare in terms of private goods [12, 5]. Lexmin allocation was developed as a fairness notion. [22] has also showed that lexmin allocations can be made to be envy free upto any good. However [18] was among the first to look into fair and efficient division of public goods. While [18] focusses on as fairness measures, [15] focusses on core as a fairness notion for fair and efficient division of indivisible public goods. [15] were the first to estimate core in an indivisible setting.

Another important direction of research is to develop algorithms for finding out these allocations. The problem is well studied for PRIVATEGOODS. [14, 6] has shown PRIVATEMNW is $\mathcal{NP}$-hard for $N = 2$ agents, but the problem becomes solvable in polynomial time if the valuations are binary [14, 6].

Using the reduction from PRIVATEGOODS to PUBLICGOODS, the authors of [18] show PUBLICMNW is $\mathcal{NP}$-hard for $k < n$. The authors also show PUBLICMNW is $\mathcal{NP}$-hard for any k and even for binary valuation which is in contrast to the PRIVATEMNW case. Finally using the reduction from PUBLICGOODS to PUBLICDECISION, the authors of [18] show DECISIONMNW is $\mathcal{NP}$-hard even for binary valuations.

4 Main Results of [18]

[18] addresses three main questions which we shall discuss breifly in the next three subsections, these three questions are as follows:

- Relating the three problem setting PRIVATEGOODS, PUBLICGOODS, and PUBLICDECISION
- Fairness and Effitiency guarantees in PUBLICGOODS
- Computing the computional complexities of the MNW and Lexmin allocations for PUBLICGOODS

4.1 Relating the three problem setting PRIVATEGOODS, PUBLICGOODS, and PUBLICDECISION

There doesn't seem to be connections between the three problem setting on first glance but the authors of [18] show there are polynomial time reductions between the MNW and lexmin allocation problems in the three problem settings.

Theorem. PRIVATEMNW polynomial-time reduces to PUBLICMNW

Proof Sketch.

Given an instance $\mathcal{I}$ of PRIVATEGOODS $\mathcal{I} = (\mathcal{A} = [n], \mathcal{G} = [m], V)$ construct an instance $\mathcal{I}$' of PUBLICGOODS $\mathcal{I}$' $= (\mathcal{A}' = [m + n], \mathcal{G}' = [m.n], k = m, \mathcal{V}')$

Create agents in $\mathcal{I}$' corresponding to the agents in $\mathcal{A}$. Then add m dummy agents, one corresponding to each private good.

Now introduce m.n public goods by making n copies of each good in $\mathcal{G}$. set k as the number of private goods

Define the valuations of i in $\mathcal{A}'$ as follows

$$v'_i(j_l) = \begin{cases} v_{ij} & \text{if } i = l \text{ and } i \in [n] \\ 1 & \text{if } i = n + j \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

This essentially refers to the situation where each agent i values the ith copy of every good j at what they valued them at the private good setting, and values all the other goods at zero. Each dummy agent j which is created corresponding to the jth private good values all public good copies of jth good at value 1 and 0 otherwise.

Now the reduction follows as:

Suppose MNW allocation of $\mathcal{I}$' has positive nash welfare, so each agent gets positive utility. Now note that the jth dummy agent only values copies of the jth good and since they have positive utility so they get atleast one copy of the jth good.

But, we have $k = m$, so only m public goods can be selected in an allocation and there are m dummy agent. So by pigeon hole principle exactly one copy of each good is selected in the allocation.

Now from an allocation x' of $\mathcal{I}$' we can construct an allocation x of I. Observe that the ith agent only values the ith copy of goods. Thus x can be constructed as $x_i = j \in G : j_i \in x'$. Clearly then $v_i(x_i) = v'_i(x')$ and every dummy agent d gets value $v'_d(x') = 1$. Thus product of utilities of agents in x = product of utilities of agents in x'. The same argument holds in the other direction by reversing the logic. i.e, from an private goods instance and an allocation of private goods, we can create a public goods instance and an allocation of public goods with the same product of utilities (nash product).

However the above argument won't hold if the maximumn nash welfare is zero. This can be handled by doubling the number of agents. Also, it is easy to observe the reduction is polynomial time.

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Theorem. PublicMNW polynomial-time reduces to DecisionMNW

PROOF SKETCH :

Given an instance $\mathcal{I}$of PUBLICGOODS $\mathcal{I}$= $(\mathcal{A} = [n], \mathcal{G} = [m], k, \{v_i\})$ construct an instance I' of PUBLICDECISIONS $\mathcal{I}$' $= (\mathcal{A}, \mathcal{G}, \{G_j\}_{j \in \mathcal{G}}, \{v_i\}_{i \in \mathcal{A}})$

When $k = m$ select all the m goods.

When $n \leq k < m$ let $V = max_{i.j} v_{ij}$, create m public issue for each $j \in \mathcal{G}$ with two alternatives $c \in \{1, 2\}$ i.e, $G_j = \{(j, 1), (j, 2)\}$, Define $\mathcal{A}' = [n + mT], T = [2mnlogmV]$ where the first n agents of $\mathcal{I}$' are same as the n agents (who values alternative 1 at v_{ij} in I and the next mT are of two types: $\{n + 1, ...n + kT\}$ of type A (who values only alternative 1) and rest of type B (who values only alternative 2).

Suppose x' be an allocation of $\mathcal{I}$' and x of $\mathcal{I}$. let S_1 be the set of issies j with decision 1 in x' and suppose there are k' such issues. Note that:

$$NW(x') = (\prod_{i \in [n]} v'_i(x').(k')^{kT}.(m - k')^{(m-k)T})^{\frac{1}{n+mT}}$$

let $x = S_1 \in \mathcal{G}$ be the set of public goods that corresponds to the decision $(j, 1)$, Then, for any $i \in [n]$, $v_i(x) = v'_i(x')$ as $v'_i(j, 2) = 0 \ \forall j \in [m]$. So,

$$NW(x') = (\prod_{i \in [n]} v_i(x).(k')^{kT}.(m - k')^{(m-k)T})^{\frac{1}{n+mT}} = (NW(x)(k')^{kT}.(m - k')^{(m-k)T})^{\frac{1}{n+mT}}$$

Now note that the nash product for an instance of PublicGoods with $k = l$, W_l is increasing in l and are bounded by $(mV)^n$. But $k \geq n$ where $l \leq m$. Define $g(a) = a * k(m - a)^{m-k}$.

Note $G_1 = k^k(m-k)^{m-k}$ and $G_2 - max(g(k-1), g(k+1))$ are the maximum two values that g can attain. Further, it can be shown $G_1^T > W_m * G_2^T$. (This can be done by expanding $log\ g(k) - log\ g(k-1) \geq 1/2m$ and $log\ g(k) - log\ g(k+1) \geq 1/2m$ giving $T(log\ G_1 - log\ G_2) > 1/2m * (2mnlogmV) \geq logW_m$ and finally using this, $W_k.g(k)^T \geq G_1^T > W_m.G_2^T \geq W_{k'}.g(k')^T$. Hence $W_{k'}.g(k')^T$ is maximises ar $k' = k$. But,

$$NW(x') = (W_{k'} g(k')^T)^{\frac{1}{n+mT}}$$

.

So $|x| = k$, and x maximises te NW among all allocations of $\mathcal{I}$cardinality constarint. and is hence MNW allocation in I. Again clearly its a polynomial time reduction.

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Theorem. PRIVATELEX polynomial-time reduces to PUBLICLEX which in turn polynomial-time reduces to DECISIONLEX

Proof Sketch.

The proofs follows from the same strategy used in the earlier two theorems.

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4.2 Fairness and Efficiency guarantees in PUBLICGOODS

The authors of [18] tried to understand how fair and efficient are the MNW and lexmin allocations in the PUBLICGOODS setting. In this regard the authors present the following results.

Theroem. When valuations are additive and monotone the following holds in a PUBLICGOODS instance.

- Any allocation that satisfies RRS also satisfies Prop1.
- Any allocation that is $\alpha-$RRS is also $\alpha.\frac{n}{2n-1}$-Prop. Further, when $n|k$, $\alpha-$RRS implies $\alpha-$Prop. This even holds when the valuations are sub-additive.
- Any allocation that satisfies $\alpha-$Prop also satisfies $\frac{\alpha}{n}-$RRS
- Any allocation that satisfy Prop1 need not satisfy $\alpha-$Prop or $\alpha-$RRS

The proofs follow from definitions. For the sake of brevity we skip the proof for these results.

Theroem. The MNW allocations for PUBLICGOODS satisfy Prop1, Pareto-Optimality, and $1/n-$RRS. Further when $k \geq n$, MNW allocation satisfies $\frac{1}{2n-1}-$Prop.

Proof Sketch:

Pareto-Optimality. Suppose MNW allocations do not satisfy pareto optimality, this would mean one of agent could get a strictly higher value keeping the values of other agents non decreasing. Consider two cases:

- MNW value $\neq 0$: Then we can get an allocation whose NW is greater. Contradiction.
- MNW value $= 0$: If the value increase holds for an agent with non zero value initially, the nash product over these agents increases, contradiction. Else, if the value increases for an agent with zero value initially then the number of agents with non zero values increases, again a contradiction to optimality of MNW.

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$\mathbf{1/n-}$**RRS.** This can be shown by the following series of steps:

- Assume there exists a MNW allocation x which doesn't satisfy $1/n-$RRS.
- Then, for some agent i, $v_i(x) < (1/n)RRS_i$.
- Order the goods according to i's valuation, such that $v_i(g_r) \geq v_i(g_s)\ \forall 1 \leq r \leq s \leq m$. Let $p = \lfloor k/n \rfloor$, if $k < n \implies p = 0, RRS_i = 0$; for $k \geq n$, the $RRS_i = v_i(g1, ..., gp)$.

- Scale the valuations so that for every agent i, $v_i(x) = 1$, which implies $RRS_i > n$
- Order the goods in x according to i's valuation: let $x = (j_1, j_2, \ldots, j_k)$, such that $v_i(j_r) \geq v_i(j_s)\ \forall 1 \leq r \leq s \leq k$. Define for $r \in [p], S_r = j_{rnn+1}, \ldots, j_{rn}$, and $g'_r = argmin_{j \in S_r} \sum_{h \in A' \setminus \{i\}} v_{hj}$
- Construct x' by removing and adding goods according to specific conditions, ensuring that $g_1, \ldots, g_p \in x'$ where $g_p = \lfloor k/n \rfloor$ which ensures $v_i(x') \geq RRS_i > n$
- Finally it can be shown $NW(x')^n = NW(x)^n$ by expanding the NW. This contradicts x is MNW

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Prop1. Directly by previous theorem.

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$\mathbf{k \geq n \implies \frac{1}{2n-1}}$**−Prop.** By previous theorem, Any allocation that is $\alpha-$RRS is also $\alpha.\frac{n}{2n-1}$-Prop. Take $\alpha = 1/n$

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Theroem. The Lexmin allocations satisfy Prop1, Pareto-Optimality, and RRS. Further when $k \geq n$, lexmin allocation satisfies $\frac{n}{2n-1}-$Prop.

The proof is similar to the earlier theorem and is skipped for brevity.

4.3 Computing the computational complexities of the MNW and Lexmin allocations for PUBLICGOODS

The problem for computational complexity of PRIVATEGOODS are well studied in literature. [14, 6] has shown PRIVATEMNW is $\mathcal{NP}$-hard for $N = 2$ agents, but the problem becomes solvable in polynomial time if the valuations are binary [14, 6]. Using the reduction from PRIVATEGOODS to PUBLICGOODS show PUBLICMNW is $\mathcal{NP}$-hard for $k < n$. The following theorems expands on the computational complexity for PUBLICMNW and PUBLICLEX.

Definitiion. Exact Regular Set Packing (ERSP) problem. Given n elements in $X = (x_1, x_2, ..., x_{n)}$ and a family of subsets of X, $F = \{F_1, ..F_m\}$ with $|F_j| = d$, the problem is to compute a subfamily $F' \subseteq F, |F'| = r$ such that $\forall F_i \neq F_j \in F', F_i \cap F_j\ \{\}$. The ERSP problem is NP-hard [16].

Theorem. PUBLICMNW and PUBLICLEX is NP-hard, even when all valuations are binary

Proof Sketch. The authors of [18] proposes a reduction of the PUBLICMNW to the ERSP problem as follows: Let $I = (X, F, d, r)$ be an instance of ERSP. Construct a PUBLICGOODS instance $I' = A, G, k, v_{ii \in A}, T$. Let $A = 1, 2, \ldots, n$, $G = g_1, \ldots, g_m \cup d_1, \ldots, d_n$ representing $m + n$ public goods. Define the valuation as:

$$v_i(g_j) = \begin{cases} 1 & \text{if } x_i = F_j \\ 0 & \text{otherwise} \end{cases}\ v_i(d_j) = 1\ \forall d_j \in G \tag{2}$$

Set $k = r + n$ and $T = \left((n+1)d^r.n^{n-dr}\right)^{1/n}$.

The authors finally show that I can be solved the ERSP problem $\iff$ the MNW for I' is at least T.

The result for by reducing from the c-monotone SAT problem which is NP hard.

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Theorem. PublicMNW and PUBLICLEX is NP-hard, even for two agents

The proof is skipped for brevity.

Corollary. The problems DECISIONMNW and DECISIONLEX are NP-hard.

The reduction from PUBLICMNW to DECISIONMNW and from PUBLICLEX to **DecisionLex** along with the fact that PUBLICMNW and PUBLICLEX are NP hard gives the corollary.

Since the computation of MNW and lexmin allocations for PUBLICGOODS are NP-hard, [18] provides an appoximation algorithm for computing the same. The algorithm provided an $O(n)$ factor approximation to MNW and satisfies fairness properties of RRS, Prop1 when valuations are monotone and subadditive.

Theorem. There exists a pseudo polynomial time approximation for PUBLICMNW when the number of agents are constant.

Insights. The run time of the algorithm is found to be $O((m.w)^n)$ where $w = max_{i,j} v_{ij}$. This indicates when the PUBLICGOODS instance has constant number of agents and agent valutations are binary PUBLICMNW can be solvable in polynomial time.

5 My Ideas

We extend the given definition of a public good to a scenario where each good has a cost and agents have a collective budget.

Modified Definition. PUBLICGOODS-COST

A public good with cost (PUBLICGOODS-COST) instance can be defined as a tuple ($\mathcal{A}$, $\mathcal{G}$,k, $\mathcal{V}$, $\mathcal{C}$,B), where:

- $\mathcal{A} = [n]$ is a set of n agents
- $\mathcal{G} = [m]$ is a set of m goods, where each good $j \in \mathcal{G}$ has a cost $c_j \in \mathbb{R} \geq 0$
- $k \in \mathbb{Z} \geq 0$ is the maximum number of goods that can be selected
- $\mathcal{V} = (v_i)_{i \in A}$ a set of agent utility functions, where $v_i : 2^{\mathcal{G}} \rightarrow \mathbb{R} \geq 0$ is a function that maps a set of goods to the utility that agent i derives from it
- $\mathcal{C} = (c_j)_{j \in \mathcal{G}}$ is the set of costs of the goods
- $B \in \mathbb{R}_{\geq 0}$ is the collective budget of the agents

An allocation x is a subset of $\mathcal{G}$ of size at most k, ($|x| \leq k$), giving agent i a utility of $v_i(x)$, subject to the budget constraint $\sum_{j \in x} c_j \leq B$.

This model reflects the fact that the agents want to find an allocation or find a set of goods that maximizes their utility but subject to the constraint that the total cost of the selected goods do not exceed a collective budget.

Example. This model has wide range of applications, here we give two such examples.

- ($k < n$): Consider a Public Transit System where there are n commuters and m origin-destination pairs for routes. Each route has a different cost associated (some are longer and need more fuel). Suppose the transit authority issues tickets at a different amount of money from each a set of OD pairs at an amount c_j making the total budget as B. The objective is to maximize the total utility of the commuters, which may depend on several factors such as the travel time, safety or ease of the travel. The company needs to open k OD pairs in a city. Clearly the model can be defined in a PUBLICGOODS-COST setting.
- ($k > n$) Consider another instance in a library with n people subscribed want to purchase k books. Each book costs c_j and the total amount that can be spent must fall below the coolective subscription fees of the library B. Each user gets an utility based on their preferences.

Allocation. The MNW allocation for this problem can be defined as follows:

$$x^* = argmax_{x \subseteq G, |x| \leq k, \sum_{j \in x} c_j \leq B} \prod_{i \in A} v_i(x)$$

Theorem. The MNW allocations for PUBLICGOODS-COST is Pareto Optimal

Proof. See Appendix - I

Fairness. $\alpha - Prop1$ for the MNW allocation can be defined as follows. An allocation x is said to be α-Prop1 if $\forall i \in \mathcal{A}$, there exists a good $g \in x$ in the allocation and another good $g' \in \mathcal{G}$ perhabs not in the allocation such that swapping g with g' in the allocation x results in a utility of at least $\alpha \cdot Prop_i$ for agent i, while still satisfying the budget constraint.

$$\exists g \in x, g' \in G \text{ with} \sum_{j \in (x-g) \cup g'} c_j \leq B$$

such that

$$v_i((x-g) \cup g') \geq \alpha.Prop_i \;\forall i$$

$\alpha - Prop1$ doesn't balance the trade off between the valuations and cost of goods, hence we can define a new fairness notion that ensures the difference between the total value derived from the goods by a group of agents and the total cost of goods is divided fairly among the agents of that group which is a stronger fairness notion. Let's define this fairness notion as Cost Adjusted Proportionality or CAP. We say an allocation x is CAP if:

$$\frac{v_i(x)}{\sum_{j \in x} c_j} \geq \beta \frac{Prop_i}{\sum_{j \in G} c_j}$$

We can define an allocation that is effitient and fair in terms of CAP, we call this allocation a BUDGETALLOCATION. Suppose such an allocation is x^*. We define BUDGETALLOCATION as a multiobjective optimisation:

Maximise product of utilities of all agents subject to budget constraints

$$max_x \prod_{i \in A} v_i(x) \text{subject to} \sum_{j \in x} c_j \leq B$$

Ensure CAP fairness for all agents

$$\frac{v_i(x)}{\sum_{j \in x} c_j} \geq \beta \frac{Prop_i}{\sum_{j \in G} c_j} \;\forall i \in A$$

Algorithm. Green Heuristic Algorithm for finding a "good" allocation PUBLICGOODS-COST

We provide a greedy heuristic for finding the allocations. Refer to Appendix - I for the same.

Theorem. The MNW and lexmin allocations for PUBLICGOODS-COST is NP hard

Proof. See Appendix - I

6 Concluding Remarks

In this report we studied fair and efficient allocation of indivisible public goods. We mainly looked into the results of [18] which mainly focuses on allocation public goods scenario of selecting k out of m public goods with n agents in fair and efficient manner. [18] showed the maximum nash welfare and lexmin allocation for the setting are efficient and fair, and that computing them is NP hard. The authors also presented polynomial time reductions between the MNW and lexmin allocations in the private good, public good and public decision setting and finally provided an approximation algorithm for computing MNW and lexmin allocations for public goods.

We extended the Public goods formulation to a Public goods with cost instance, where each public good has some costs and there are budget constraints. We showed the MNW allocation for the problem and proposed a new fairness notion and an allocation based on that as a multi objective optimisation problem. We showed solving the MNW allocation for this problem is NP hard. We also proposed a naive greedy heuristic to get a good allocation for the problem. There are several future directions for this setting – investigating if the reductions to public decisions will hold and if the proposed MNW allocation in this setting fair.

Other future directions include developing a polynomial time prop1 and pareto-optinal allocation for public goods and developing constsnt factor approximation algorithm for public MNW even for special cases.

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A Appendix - I

Theorem. The computation of MNW allocations for PUBLICGOODS-COST is NP Hard.

Proof.

We show this by doing a reduction from PUBLICGOODS-COST problem with budget constraints to the PUBLICGOODS problem without budget constraints. We already know the MNW allocations for PUBLICGOODS problem without budget constraints is NP hard. We will show that an instance of PUBLICGOODS problem can be transformed into an instance of the PUBLICGOODS-COST problem, demonstrating that the PUBLICGOODS-COST problem is at least as hard as the PUBLICGOODS problem.

Consider an instance of the PUBLICGOODS problem, described by the tuple $(\mathcal{A}, \mathcal{G},k, \mathcal{V})$, where:

$\mathcal{A} = [n]$ is a set of n agents $\mathcal{G} = [m]$ is a set of m goods $k \in \mathbb{Z}_{\geq 0}$ is the maximum number of goods that can be selected $\mathcal{V} = v_i i \in A$ a set of agent utility functions, where $v_i : 2^{\mathcal{G}} \to \mathbb{R} \geq 0$ is a function that maps a set of goods to the utility that agent i derives from it

We can transform this instance into an instance of the PUBLICGOODS-COST problem by introducing budget constraints. Define the tuple $(\mathcal{A}, \mathcal{G},k, \mathcal{V}, \mathcal{C},B)$, where:

$\mathcal{A}, \mathcal{G},k, \mathcal{V}$ remain the same as in the original PUBLICGOODS problem $\mathcal{C} = c_{j\, j \in \mathcal{G}}$ is the set of costs of the goods, where each cost $c_j = 1$ for all $j \in \mathcal{G}$, that is cost of each good is 1. Set $B = k$. That is the collective budget of the agents is set equal to the maximum number of goods that can be selected.

Now, observe that any allocation of goods in the PUBLICGOODS problem can also be an allocation in the PUBLICGOODS-COST problem, since:

- The total cost of any allocation in the PUBLICGOODS problem is at most k (because there are at most k goods in the allocation and each good has a cost of 1).
- The budget constraint in the PUBLICGOODS-COST problem is set to $B = k$, which means that any allocation in the PUBLICGOODS problem satisfies the budget constraint in the PUBLICGOODS-COST problem.

Therefore, any solution to the PUBLICGOODS problem is also a valid solution for the corresponding PUBLICGOODS-COST problem. This shows that the PUBLICGOODS-COST problem is NP-hard.

■

Theorem. The MNW allocations for PUBLICGOODS-COST is Pareto Optimal

Proof. Suppose MNW allocations for PUBLICGOODS-COST do not satisfy pareto optimality, this would mean one of agent could get a strictly higher value keeping the values of other agents non decreasing. Consider two cases:

- MNW value $\neq 0$: Then we can get an allocation whose NW is greater. Contradiction.
- MNW value $= 0$: If the value increase holds for an agent with non zero value initially, the nash product over these agents increases, contradiction. Else, if the value increases for an

agent with zero value initially then the number of agents with non zero values increases, again a contradiction to optimality of MNW.

Note that the whole analysis falls through even with budget constraints.

■

Algorithm.

Given below is a greedy algorithm for allocating goods for the problem. The algorithm uses another algorithm to find optimal MNW allocation without budget constraints as a subroutine.

Algorithm 1: Greedy Algorithm for Public Goods with Budget Constraints

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Input: Set of agents $\mathcal{A}$, set of goods $\mathcal{G}$, maximum number of goods k, agent utility functions v_i, goods costs c_j, collective budget B

Output: Final allocation x

Initialize an empty set for the final allocation x and a remaining budget $B_r = B$

Sort the goods $\mathcal{G}$ in decreasing order of their utility-to-cost ratio: $\frac{v_i(g)}{c_g}$, where $v_i(g)$ is the utility derived by agent i from good g, and c_g is the cost of good g

for *each good g in the sorted list of goods* **do**

    if *adding g to the allocation x does not violate the budget constraint and the size constraint, i.e., $\sum_{j \in x \cup g} c_j \leq B_r$ and $|x^* \cup g| \leq k$* **then**

        Add g to the allocation x; Update the remaining budget: $B_r = B_r - c_g$;

return the final allocation x;
